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On the Mordell-Weil Lattice of the Elliptic Curve $y^2 = x^3 + t^m + 1$. III

by

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1. Introduction

In [U1] and [U2], we consider the Mordell-Weil lattice of the elliptic curve

$$E^{(m)} : y^2 = x^3 + t^m + 1$$

defined over $K = k(t)$, where k is an algebraically closed field of characteristic 0. We denote by L_m the Mordell-Weil lattice $E^{(m)}(K)$. The main result of [U1] is that all L_m are described by $L_9, L_{12}, L_{18}, L_{24}, L_{30}, L_{60}$ and well known root lattices. In [U2], we study the lattices $L_9, L_{12}, L_{18}, L_{24}, L_{30}, L_{60}$ and determine the lattice structures of L_9 and L_{12}. But the method we use in [U2] is computer search and we need too much time to determine the lattice structures for other cases. In this paper, we first show two lemmas. Lemma 1 (resp. Lemma 2) states that if a sublattice satisfies some conditions, then $2P$ (resp. $3P$) is in the sublattice for every lattice point P. Next we give a sublattice which satisfies the conditions of one of the two lemmas for each case. In particular, for the case L_{30}, we give a sublattice which satisfies the conditions of both of the two lemmas. So this sublattice is precisely the lattice L_{30}. For the case L_{18}, we give a sublattice which satisfies the conditions of Lemma 1. Moreover, in a similar method to the lemmas, we can show that this sublattice is precisely the lattice L_{18}.

We use the same notation as in [U1] and [U2]. We denote by $L(c)$ the lattice whose pairing is c times of the pairing of the lattice L. So we denote the lattice $E^{(ab)}(k(t^b))$ by $L_a(b)$. This is a sublattice of L_{ab}. For sublattices N_1 and N_2 of L, $N_1 + N_2$ is the sublattice of L generated by N_1 and N_2. If the pairing $\langle P_1, P_2\rangle = 0$ for all $P_1 \in N_1$ and $P_2 \in N_2$, we say that $N_1 + N_2$ is a direct sum and denote it by $N_1 \oplus N_2$. A_2^*, D_4^*, E_6^* are the dual lattices of the root lattices A_2, D_4, E_6 respectively and E_8 is the unique positive-definite even unimodular lattice of rank 8 (cf. [C-S, Ch.4]).

If $P = (x(t), y(t)) \in L_{6n}$, then we have

$$\widetilde{P} = (t^{2n}x(1/t), t^{3n}y(1/t)) \in L_{6n}\,. \tag{1.1}$$

For a sublattice N of L_{6n}, we denote

$$\widetilde{N} = \{\widetilde{P} \mid P \in N\}\,. \tag{1.2}$$

The map defined by

$$\iota : P \longmapsto \widetilde{P} \tag{1.3}$$

is an automorphism on L_{6n} such that $\iota^2(P) = P$. We have

$$\widetilde{N} \cong N \tag{1.4}$$

as lattices.

Let C_n be the elliptic curve defined by

$$C_n : y^2 = x^3 + t^{n+1} + t\,. \tag{1.5}$$

Then

$$M = \{(t^4 x(1/t^3), t^6 y(1/t^3)) \mid (x(t), y(t)) \in C_3(k(t))\} \tag{1.6}$$

and

$$M_n = \{(x(t^6)/t^2, y(t^6)/t^3) \mid (x(t), y(t)) \in C_n(k(t))\} \tag{1.7}$$

are sublattices of L_9 and L_{6n} respectively. In section 3, we show that these sublattices are primitive. We say that N is a primitive sublattice of L if the quotient L/N is torsion-free. For a sublattice N of L, we denote the primitive closure of N by $\overline{N}$. That is

$$\overline{N} = \{P \in L \mid mP \in N \text{ for some natural number } m\}\,. \tag{1.8}$$

The main results are as follows:

$$
\begin{array}{llll}
3L_9 \subset & L_3(3) \oplus M & \subset L_9 & \text{(Proposition 1)} \\
3L_{12} \subset & L_4(3) + L_6(2) + \widetilde{L_4(3)} & \subset L_{12} & \text{(Proposition 2)} \\
L_{18} = & L_9(2) + L_6(3) + \widetilde{L_9(2)} & & \text{(Theorem 1)} \\
2L_{24} \subset & L_{12}(2) \oplus M_4 & \subset L_{24} & \text{(Proposition 4)} \\
L_{30} = & L_5(6) \oplus L_6(5) \oplus \widetilde{L_5(6)} & & \text{(Theorem 2)} \\
L_{60} = & L_5(12) \oplus \widetilde{L_5(12)} \oplus \overline{L_{12}(5) \oplus M_{10}} & & \text{(Proposition 7)}
\end{array}
$$

We know that

$$L_3 \cong D_4^*,\ L_4 \cong E_6^*,\ L_5 \cong L_6 \cong E_8,\ M \cong E_6^*(3),\ M_4 \cong E_8(6) \tag{1.9}$$

([O-S], [U1]).

In section 5, using the data of [U2], we investigate L_9, L_{12} and L_{18} more explicitly. In [U2], we use

$$E_-^{(m)} : y^2 = x^3 + t^m - 1$$

in place of

$$E^{(m)} : y^2 = x^3 + t^m + 1\,,$$

and we show

(1.10) $$L_m^{(p_m)} := E_-^{(m)}(\overline{\mathbb{F}_{p_m}}(t)) = E_-^{(m)}(\mathbb{F}_{p_m}(t)) \quad (m = 9, 12, 18, 24, 30)\,.$$

Here

(1.11) $$p_9 = 433\,, \quad p_{12} = 397\,, \quad p_{18} = 433\,, \quad p_{24} = 1801\,, \quad p_{30} = 25261$$

are splitting prime numbers. $\mathbb{F}_{p_m}$ is the finite field of p_m elements and $\overline{\mathbb{F}_{p_m}}$ is its algebraic closure. We could not find a splitting prime number for the case $m = 60$. We will mention it in subsection 4.6. We can calculate the height pairings in $L_m^{(p_m)}$. Taking the height pairings with independent points, we can treat all points as vectors. We have explicit generators of $L_9^{(433)}$ and $L_{12}^{(397)}$ ([U2, Theorem 2, 3]).

According to Shioda, it is known that

(1.12) $$L_m^{(p_m)} \cong L_m$$

as lattices.

For the general theory of Mordell-Weil lattices, we refer to [S1] and [S3].

I would like to thank Professor Shioda and Professor Kuwata for their useful advice. In particular, the results about C_{10} are obtained by following the advice of Professor Shioda.

2. Two lemmas

NOTATION. Throughout the paper, the following notation will be used.

We define $\rho \in \mathrm{Gal}(k(t)/k(t^6))$ by

(2.1) $$\rho : t \to \zeta_6 t\,,$$

where ζ_6 is a primitive 6-th root of 1. We define $\tau = \rho^3 \in \mathrm{Gal}(k(t)/k(t^2))$ and $\sigma = \rho^2 \in \mathrm{Gal}(k(t)/k(t^3))$

(2.2) $$\tau : t \to -t$$

(2.3) $$\sigma : t \to \omega t \quad (\omega = {\zeta_6}^2)\,.$$

For a point $P = (x, y)$, we denote

(2.4) $$\omega P = (\omega x, y)\,.$$

LEMMA 1. *Assume that N is a primitive sublattice of L_{2n} such that $P^\tau \in N$ for all $P \in N$. If $L_n(2) + N$ is a sublattice of finite index in L_{2n}, then $2P \in L_n(2) + N$ for all $P \in L_{2n}$.*

Proof. Let $P \in L_{2n}$. Since $L_n(2) + N$ is a sublattice of finite index in L_{2n}, there exist a natural number m, $Q \in L_n(2)$ and $R \in N$ such that

$$mP = Q + R\,.$$

Then

$$(mP)^\tau = Q^\tau + R^\tau = Q + R^\tau$$

and

$$m(P - P^{\tau}) = mP - (mP)^{\tau} = R - R^{\tau} \in N\,.$$

Since N is a primitive sublattice of L_{2n}, we have

$$P - P^{\tau} \in N\,.$$

On the other hand, since

$$(P + P^{\tau})^{\tau} = P + P^{\tau},$$

we have

$$P + P^{\tau} \in L_n(2)\,.$$

Then

$$2P = (P + P^{\tau}) + (P - P^{\tau}) \in L_n(2) + N\,.$$

q.e.d.

LEMMA 2. *Assume that N is a primitive sublattice of L_{3n} such that $P^{\sigma} \in N$ for all $P \in N$. If $L_n(3) + N$ is a sublattice of finite index in L_{3n}, then $3P \in L_n(3) + N$ for all $P \in L_{3n}$.*

Proof. Let $P \in L_{3n}$. Since $L_n(3) + N$ is a sublattice of finite index in L_{3n}, there exist a natural number m, $Q \in L_n(3)$ and $R \in N$ such that

$$mP = Q + R\,.$$

Then

$$(mP)^{\sigma} = Q^{\sigma} + R^{\sigma} = Q + R^{\sigma}$$

and

$$m(P - P^{\sigma}) = mP - (mP)^{\sigma} = R - R^{\sigma} \in N\,.$$

Since N is a primitive sublattice of L_{3n}, we have

$$P - P^{\sigma} \in N\,.$$

Then

$$P - P^{\sigma^2} = P - P^{\sigma} + P^{\sigma} - P^{\sigma^2} = P - P^{\sigma} + (P - P^{\sigma})^{\sigma} \in N\,.$$

On the other hand, since

$$(P + P^{\sigma} + P^{\sigma^2})^{\sigma} = P + P^{\sigma} + P^{\sigma^2},$$

we have

$$P + P^{\sigma} + P^{\sigma^2} \in L_n(3)\,.$$

Then

$$3P = (P + P^{\sigma} + P^{\sigma^2}) + (P - P^{\sigma}) + (P - P^{\sigma^2}) \in L_n(3) + N\,.$$

q.e.d.

3. The sublattices M and M_n

In this section, we show that M and M_n (defined by (1.6) and (1.7)) are primitive sublattces of L_9 and L_{6n} respectively.

LEMMA 3. *Let $P \in L_9$. Then $P^\sigma = \omega P$ if and only if $P \in M$.*

Proof. Let $P = (x(t), y(t))$ and assume that $P^\sigma = \omega P$. Since

$$x(\omega t) = \omega x(t)\,, \quad y(\omega t) = y(t)\,,$$

$x(t)/t$ and $y(t)$ are $\sigma-$invariant. So there are rational functions $x_1(t)$ and $y_1(t)$ such that

$$x(t)/t = x_1(t^3)\,, \quad y(t) = y_1(t^3)\,.$$

Substituting them into the defining equation of $E^{(9)}$, we have

$$y_1(t^3)^2 = t^3 x_1(t^3)^3 + t^9 + 1\,.$$

This means that $x_1(t)$ and $y_1(t)$ satisfy the equation

$$y_1(t)^2 = t x_1(t)^3 + t^3 + 1\,.$$

If we let

$$(x_2(t), y_2(t)) = (t x_1(1/t), t^2 y_1(1/t))\,,$$

then we have

$$(x_2(t), y_2(t)) \in C_3(k(t))$$

and

$$P = (t^4 x_2(1/t^3), t^6 y_2(1/t^3)) \in M\,.$$

The converse is clear. *q.e.d.*

LEMMA 4. *M is a primitive sublattice of L_9.*

Proof. We assume that $mP \in M$. Then

$$(mP)^\sigma = \omega(mP)\,.$$

So we have

$$m(P^\sigma - \omega P) = O\,.$$

Since L_9 is torsion-free, we have

$$P^\sigma - \omega P = O$$

and

$$P^\sigma = \omega P\,.$$

By Lemma 3, this means that $P \in M$. So M is a primitive sublattices of L_9. *q.e.d.*

LEMMA 5. *Let $P \in L_{6n}$. Then $P^\rho = -\omega^2 P$ if and only if $P \in M_n$.*

Proof. Let $P = (x(t), y(t))$ and assume that $P^\rho = -\omega^2 P$. Since

$$x(\zeta_6 t) = \omega^2 x(t)\,, \quad y(\zeta_6 t) = -y(t)\,,$$

$x(t)/t^4$ and $y(t)/t^3$ are ρ-invariant. So there are rational functions $x_1(t)$ and $y_1(t)$ such that

$$x(t)/t^4 = x_1(t^6)\,, \quad y(t)/t^3 = y_1(t^6)\,.$$

Substituting them into the defining equation of $E^{(6n)}$, we have

$$t^6 y_1(t^6)^2 = t^{12} x_1(t^6)^3 + t^{6n} + 1\,.$$

This means that $x_1(t)$ and $y_1(t)$ satisfy the equation

$$ty_1(t)^2 = t^2x_1(t)^3 + t^n + 1\,.$$

If we let

$$(x_2(t), y_2(t)) = (tx_1(t), ty_1(t))\,,$$

then we have

$$(x_2(t), y_2(t)) \in C_n(k(t))$$

and

$$P = (t^4x_2(t^6)/t^6, t^3y_2(t^6)/t^6) = (x_2(t^6)/t^2, y_2(t^6)/t^3) \in M_n\,.$$

The converse is clear. *q.e.d.*

LEMMA 6. *M_n is a primitive sublattice of L_{6n}.*

Proof. We assume that $mP \in M_n$. Then

$$(mP)^\rho = -\omega^2(mP)\,.$$

In the same way as the proof of Lemma 4, we can show that

$$P^\rho = -\omega^2P\,.$$

By Lemma 5, this means that $P \in M_n$. So M_n is a primitive sublattices of L_{6n}. *q.e.d.*

4. Main results

In this section, we use the following lemma from [U1].

LEMMA 7 [U1, Lemma 3.3]. *If $(b, c) = 1$, then $L_{ab}(c) + L_{ac}(b)$ is a primitive sublattice of L_{abc}.*

Though we did not mention it explicitly in the proof of [U1, Theorem 2], we have

$$L_{10} = L_5(2) \oplus L_2(5) \tag{4.1}$$

$$L_{15} = L_5(3) \oplus L_3(5) \tag{4.2}$$

$$L_{20} = L_5(4) \oplus L_4(5) \tag{4.3}$$

and so on, by using the following fact.

LEMMA 8. *Assume that N is a sublattice of L_{ab} such that $P^\beta \in N$ for all $P \in N$ and $\beta \in \mathrm{Gal}(k(t)/k(t^b))$. If $L_a(b) \cap N = \{O\}$, then $L_a(b)$ and N are orthogonal.*

Proof. Let $P \in L_a(b)$ and $Q \in N$.

$$\begin{aligned}\langle P, Q\rangle &= \langle P^\beta, Q^\beta\rangle = \langle P, Q^\beta\rangle \\ &= \langle P, Q^{\beta^2}\rangle \\ &\ \ \vdots \\ &= \langle P, Q^{\beta^{b-1}}\rangle\,.\end{aligned}$$

Since $Q + Q^{\beta} + \cdots + Q^{\beta^{b-1}} \in L_a(b) \cap N = \{O\}$,

$$\begin{aligned} b\langle P, Q\rangle &= \langle P, Q\rangle + \langle P, Q^{\beta}\rangle + \cdots + \langle P, Q^{\beta^{b-1}}\rangle \\ &= \langle P, Q + Q^{\beta} + \cdots + Q^{\beta^{b-1}}\rangle = \langle P, O\rangle = 0\,. \end{aligned}$$

So we have $\langle P, Q\rangle = 0$. *q.e.d.*

By this lemma, we can also see that

$$L_5(12) + L_{12}(5) = L_5(12) \oplus L_{12}(5)\,. \tag{4.4}$$

4.1. L_9

$L_3(3) \oplus M$ is a sublattice of finite index in L_9 ([U2, Lemma 2.3]).

PROPOSITION 1. $3L_9 \subset L_3(3) \oplus M \subset L_9$

Proof. By Lemma 4, M is a primitive sublattice of L_9. We can verify that $P^{\sigma} \in M$ for all $P \in M$. Since $L_3(3)\oplus M$ is a sublattice of finite index in L_9, we have this proposition by Lemma 2. *q.e.d.*

4.2. L_{12}

The corresponding elliptic surface is K3 and

$$E^{(12)} : y^2 = x^3 + t^{12} + 1$$

is $F^{(6)}$ in Example (1) of [S5].

$L_4(3) + L_6(2) + \widetilde{L_4(3)}$ is a sublattice of finite index in L_{12} ([U2, Lemma 3.2, Remark 2]).

PROPOSITION 2. $3L_{12} \subset L_4(3) + L_6(2) + \widetilde{L_4(3)} \subset L_{12}$.

Proof. $N := L_6(2) + \widetilde{L_4(3)}$ is isomorphic to $\widetilde{N} = \widetilde{L_6(2)} + L_4(3) = L_6(2) + L_4(3)$, which is a primitive sublattice of L_{12} by Lemma 7. So N is a primitive sublattice of L_{12}. Since $P^{\sigma} \in N$ for all $P \in N$ and $L_4(3) + L_6(2) + \widetilde{L_4(3)}$ is a sublattice of finite index in L_{12}, we have this proposition by Lemma 2. *q.e.d.*

M_2 is a sublattice of $\widetilde{L_4(3)}$ and $M_2 \cong D_4^*(6)$. So $\widetilde{M_2}$ is a sublattice of $L_4(3)$. $\widetilde{M_2} \oplus L_6(2) \oplus M_2$ is a sublattice of finite index in L_{12} and

$$\widetilde{M_2} \oplus L_6(2) \oplus M_2 \subset L_4(3) + L_6(2) + \widetilde{L_4(3)}\,. \tag{4.5}$$

By Lemma 1, we have

$$2P \in L_6(2) \oplus \overline{M_2 \oplus \widetilde{M_2}} \tag{4.6}$$

for all $P \in L_{12}$. Here $\overline{M_2 \oplus \widetilde{M_2}}$ is the primitive closure of $M_2 \oplus \widetilde{M_2}$, that is,

$$\overline{M_2 \oplus \widetilde{M_2}} = \{P \in L_{12} \mid mP \in M_2 \oplus \widetilde{M_2} \text{ for some natural number } m\}\,. \tag{4.7}$$

By Proposition 2 and (4.6), we have

$$P = 3P - 2P \in L_4(3) + L_6(2) + \widetilde{L_4(3)} + \overline{M_2 \oplus \widetilde{M_2}}$$

and

$$L_{12} = L_4(3) + L_6(2) + \widetilde{L_4(3)} + \overline{M_2 \oplus \widetilde{M_2}}\,. \tag{4.8}$$

4.3. L_{18}

$L_9(2) + L_6(3) + M_3$ is a sublattice of finite index in L_{18} and

$$L_9(2) + L_6(3) + M_3 \subset L_9(2) + L_6(3) + \widetilde{L_9(2)} \subset L_{18}$$

([U2, Lemma 4.1, Remark 3]).

PROPOSITION 3. $\quad 2L_{18} \subset L_9(2) + L_6(3) + \widetilde{L_9(2)} \subset L_{18}$

Proof. By Lemma 7, $N := L_6(3) + \widetilde{L_9(2)} \cong \widetilde{L_6(3)} + L_9(2) = L_6(3) + L_9(2)$ is a primitive sublattice of L_{18}. Since $P^\tau \in N$ for all $P \in N$ and $L_9(2) + L_6(3) + \widetilde{L_9(2)}$ is a sublattice of finite index in L_{18}, we have this proposition by Lemma 1. *q.e.d.*

LEMMA 9. *For all* $P \in L_{18}$, $\omega P - P^\sigma \in L_9(2) + L_6(3)$.

Proof. Let $P \in L_{18}$. Since $L_9(2) + L_6(3) + M_3$ is a sublattice of finite index in L_{18}, there exist a natural number m, $Q \in L_9(2) + L_6(3)$ and $R \in M_3$ such that

$$mP = Q + R\,.$$

Then

$$(mP)^\sigma = Q^\sigma + R^\sigma = Q^\sigma + \omega R$$

and

$$m(\omega P - P^\sigma) = \omega(mP) - (mP)^\sigma = \omega Q - Q^\sigma \in L_9(2) + L_6(3)\,.$$

By lemma 7, $L_9(2) + L_6(3)$ is a primitive sublattice of L_{18}. So we have

$$\omega P - P^\sigma \in L_9(2) + L_6(3)\,.$$

q.e.d.

LEMMA 10. *For all* $P \in L_{18}$, $\omega P - P^{\sigma^2} \in L_6(3) + \widetilde{L_9(2)}$.

Proof. Let $P \in L_{18}$. Since $\widetilde{M}_3 + L_6(3) + \widetilde{L_9(2)}$ is a sublattice of finite index in L_{18}, there exist a natural number m, $Q \in L_6(3) + \widetilde{L_9(2)}$ and $R \in \widetilde{M}_3$ such that

$$mP = Q + R\,.$$

If $\widetilde{R} = (x(t^6)/t^2, y(t^6)/t^3)$, then

$$R = (t^8 x(1/t^6), t^{12} y(1/t^6))$$

and

$$R^\sigma = \omega^2 R\,.$$

So we have

$$(mP)^{\sigma^2} = Q^{\sigma^2} + R^{\sigma^2} = Q^{\sigma^2} + \omega R$$

and

$$m(\omega P - P^{\sigma^2}) = \omega(mP) - (mP)^{\sigma^2} = \omega Q - Q^{\sigma^2} \in L_6(3) + \widetilde{L_9(2)}\,.$$

By lemma 7, $L_6(3) + \widetilde{L_9(2)} \cong L_6(3) + L_9(2)$ is a primitive sublattice of L_{18}. So we have

$$\omega P - P^{\sigma^2} \in L_6(3) + \widetilde{L_9(2)}\,.$$

q.e.d.

THEOREM 1. $L_{18} = L_9(2) + L_6(3) + \widetilde{L_9(2)}$.

Proof. Since $P + P^\sigma + P^{\sigma^2} \in L_6(3)$, by Proposition 3, Lemma 9 and Lemma 10, we have

$$\begin{aligned} P &= (P + P^\sigma + P^{\sigma^2}) + (\omega P - P^\sigma) + (\omega P - P^{\sigma^2}) - 2\omega P \\ &\in L_9(2) + L_6(3) + \widetilde{L_9(2)}\,. \end{aligned}$$

q.e.d.

For the number of minimal sections and the determinant, see Corollary 1 in subsection 5.3.

4.4. L_{24}

$L_{12}(2) \oplus M_4$ is a sublattice of finite index in L_{24}. ([U2, Lemma 4.2]).

PROPOSITION 4. $2L_{24} \subset L_{12}(2) \oplus M_4 \subset L_{24}$.

Proof. By Lemma 6, M_4 is a primitive sublattice of L_{24}. Since $P^\tau \in M_4$ for all $P \in M_4$ and $L_{12}(2) \oplus M_4$ is a sublattice of finite index in L_{24}, we have this proposition by Lemma 1. *q.e.d.*

4.5. L_{30}

$L_5(6) \oplus L_6(5) \oplus M_5$ is a sublattice of finite index in L_{30} ([U2, Lemma 4.3]).

LEMMA 11. $M_5 = \widetilde{L_5(6)}$.

Proof. Let $P \in L_5(6)$. There are rational functions $x(t)$ and $y(t)$ such that

$$P = (x(t^6), y(t^6))\,.$$

Then we have

$$\widetilde{P} = (t^{10}x(1/t^6), t^{15}y(1/t^6)) = (t^{12}x(1/t^6)/t^2, t^{18}y(1/t^6)/t^3)\,.$$

If we let

$$(x_1(t), y_1(t)) = (t^2x(1/t), t^3y(1/t))\,,$$

then we have

$$(x_1(t), y_1(t)) \in C_5(k(t))$$

and

$$\widetilde{P} = (x_1(t^6)/t^2, y_1(t^6)/t^3) \in M_5\,.$$

Next we let $P \in M_5$. Then

$$P = (x_2(t^6)/t^2, y_2(t^6)/t^3),\ \ (x_2(t), y_2(t)) \in C_5(k(t))$$

and

$$\widetilde{P} = (t^{10}x_2(1/t^6)t^2, t^{15}y_2(1/t^6)t^3) = (t^{12}x_2(1/t^6), t^{18}y_2(1/t^6)) \in L_5(6)\,.$$

So we have $P \in \widetilde{L_5(6)}$. *q.e.d.*

PROPOSITION 5. $2L_{30} \subset L_5(6) \oplus L_6(5) \oplus \widetilde{L_5(6)} \subset L_{30}$.

Proof. By (4.2), we have $L_{15}(2) = L_5(6) \oplus L_3(10)$. Then we have

$$L_5(6) \oplus L_6(5) \oplus \widetilde{L_5(6)} = L_{15}(2) + (L_6(5) \oplus \widetilde{L_5(6)})\,.$$

By Lemma 7, $N := L_6(5) \oplus \widetilde{L_5(6)} \cong \widetilde{L_6(5)} \oplus L_5(6) = L_6(5) \oplus L_5(6)$ is a primitive sublattice of L_{30}. Since $P^\tau \in N$ for all $P \in N$ and $L_5(6) \oplus L_6(5) \oplus \widetilde{L_5(6)}$ is a sublattice of finite index in L_{30}, we have this proposition by Lemma 1. *q.e.d.*

PROPOSITION 6. $3L_{30} \subset L_5(6) \oplus L_6(5) \oplus \widetilde{L_5(6)} \subset L_{30}$.

Proof. By (4.1), we have $L_{10}(3) = L_5(6) \oplus L_2(15)$. Then we have

$$L_5(6) \oplus L_6(5) \oplus \widetilde{L_5(6)} = L_{10}(3) + (L_6(5) \oplus \widetilde{L_5(6)})\,.$$

Since $N = L_6(5) \oplus \widetilde{L_5(6)}$ is a primitive sublattice of L_{30} such that $P^\sigma \in N$ for all $P \in N$ and $L_5(6) \oplus L_6(5) \oplus \widetilde{L_5(6)}$ is a sublattice of finite index in L_{30}, we have this proposition by Lemma 2. *q.e.d.*

THEOREM 2. $L_{30} = L_5(6) \oplus L_6(5) \oplus \widetilde{L_5(6)} \cong E_8(6) \oplus E_8(5) \oplus E_8(6)$. *In particular, there are* 240 *minimal sections of norm* 10 *and the determinant is* $2^{16}3^{16}5^8$.

Proof. By Proposition 5 and Proposition 6, for all $P \in L_{30}$,

$$P = 3P - 2P \in L_5(6) \oplus L_6(5) \oplus \widetilde{L_5(6)}\,.$$

We know that $L_5 \cong E_8$ and $L_6 \cong E_8$ ([O-S], [U1]). E_8 has 240 minimal sections of norm 2 and $\det E_8 = 1$ ([C-S]). So there are 240 minimal sections of norm 10 and

$$\det L_{30} = \det(E_8(6) \oplus E_8(5) \oplus E_8(6)) = 6^8 5^8 6^8 = 2^{16}3^{16}5^8\,.$$

q.e.d.

4.6. L_{60}

$(L_{12}(5) + L_{30}(2)) \oplus M_{10}$ is a sublattice of finite index in L_{60} ([U2, Lemma 4.4]). By Theorem 2 and (4.4), we have

$$\begin{aligned}
&(L_{12}(5) + L_{30}(2)) \oplus M_{10} \\
&\quad = (L_{12}(5) + (L_5(12) \oplus L_6(10) \oplus \widetilde{L_5(12)})) \oplus M_{10} \\
&\quad = L_5(12) \oplus L_{12}(5) \oplus \widetilde{L_5(12)} \oplus M_{10}\,.
\end{aligned}$$

For a sublattice N of L_{60}, $\overline{N}$ is the primitive closure of N defined by (1.8).

PROPOSITION 7. $L_{60} = L_5(12) \oplus \widetilde{L_5(12)} \oplus \overline{L_{12}(5) \oplus M_{10}}$.

Proof. By Lemma 1, we have

$$\begin{aligned} 2P \in\ & L_{30}(2) + \overline{L_{12}(5) \oplus M_{10}} \\ =\ & L_5(12) \oplus \widetilde{L_5(12)} \oplus \overline{L_{12}(5) \oplus M_{10}} \\ \subset\ & L_5(12) \oplus \overline{\widetilde{L_5(12)} \oplus L_{12}(5) \oplus M_{10}}\,. \end{aligned} \tag{4.9}$$

By Lemma 2, we have

$$3P \in L_{20}(3) + \overline{\widetilde{L_5(12)} \oplus L_{12}(5) \oplus M_{10}}\,. \tag{4.10}$$

Since we know $L_{20}(3) = L_5(12) \oplus L_4(15)$ by (4.3),

$$\begin{aligned} 3P \in\ & (L_5(12) \oplus L_4(15)) + \overline{\widetilde{L_5(12)} \oplus L_{12}(5) \oplus M_{10}} \\ =\ & L_5(12) \oplus \overline{\widetilde{L_5(12)} \oplus L_{12}(5) \oplus M_{10}}\,. \end{aligned} \tag{4.11}$$

From (4.9) and (4.11), we obtain

$$L_{60} = L_5(12) \oplus \overline{\widetilde{L_5(12)} \oplus L_{12}(5) \oplus M_{10}}\,.$$

This means that $L_5(12)$ is a direct summand of L_{60}. Since

$$L_{60} = \widetilde{L_{60}} = \widetilde{L_5(12)} \oplus \overline{L_5(12) \oplus \widetilde{L_{12}(5)} \oplus \widetilde{M_{10}}}\,,$$

$\widetilde{L_5(12)}$ is also a direct summand of L_{60}. *q.e.d.*

<u>The sublattice M_{10}</u>

M_{10} is the lattice whose pairing is 6 times of the pairing of the lattice $C_{10}(k(t))$ and

$$C_{10} : y^2 = x^3 + t^{11} + t \tag{4.12}$$

is $F^{(5)}$ for Example (1) of [S5]. By Theorem 3.1 of [S5],

$$\det C_{10}(k(t)) = 5^4\,. \tag{4.13}$$

The rank 8 part of $C_{10}(k(t))$ is given in [CMT] as follows.
Dividing the equation of C_{10} by t^6, we have

$$(y/t^3)^2 = (x/t^2)^3 + t^5 + 1/t^5\,.$$

Letting $s = t + 1/t$,

$$(y/t^3)^2 = (x/t^2)^3 + s^5 - 5s^3 + 5s\,.$$

Let E be the elliptic curve over $k(s)$ defined by

$$E : y^2 = x^3 + s^5 - 5s^3 + 5s\,. \tag{4.14}$$

For each $P = (x(s), y(s)) \in E(k(s)) \cong E_8$ (cf. [O-S]), we have

$$y(t + 1/t)^2 = x(t + 1/t)^3 + t^5 + 1/t^5$$

and

$$Q = (t^2 x(t + 1/t), t^3 y(t + 1/t)) \in C_{10}(k(t))\,.$$

These points generate the sublattice which is isomorphic to $E_8(2)$.

Moreover, following the advice of Shioda, we have

$$y(\zeta_5 t + 1/\zeta_5 t)^2 = x(\zeta_5 t + 1/\zeta_5 t)^3 + t^5 + 1/t^5$$

and

$$R = (t^2 x(\zeta_5 t + 1/\zeta_5 t), t^3 y(\zeta_5 t + 1/\zeta_5 t)) \in C_{10}(k(t)) .$$

Here ζ_5 is a primitive 5-th root of 1. If

$$u^5 - 1 = 0 \tag{4.15}$$

can be solved in $\mathbb{F}_p$, then $\zeta_5 \in \mathbb{F}_p$.

The points of the form:

$$P = (g + as + bs^2, h + cs + ds^2 + es^3) \quad g, a, b, h, c, d, e \in k$$

generate $E(k(s))$. If we let $u = \mathrm{sp}_{s=0}(P) = g/h$, then u satisfy the fundamental algebraic equation of E:

$$u^{240} - 75600u^{228} + 1211326200u^{216} + \cdots = 0 \tag{4.16}$$

(cf. [S2, (8.23)]).

The minimal prime number p such that the equations (4.15) and (4.16) can be solved in $\mathbb{F}_p$ is 1861. In the same way as [U2], we can find 240 minimal sections of $E(\mathbb{F}_{1861}(s))$ and 8 generators of them.

$P_1 = (899s^2 + 1845s + 1166, 1355s^3 + 1395s^2 + 311s + 1719)$
$P_2 = (1028s^2 + 247s + 399, 1048s^3 + 300s^2 + 1634s + 1207)$
$P_3 = (871s^2 + 552s + 1362, 1784s^3 + 323s^2 + 455s + 718)$
$P_4 = (1727s^2 + 834s + 359, 1848s^3 + 1164s^2 + 1689s + 1160)$
$P_5 = (524s^2 + 618s + 524, 674s^3 + 1853s^2 + 1356s + 674)$
$P_6 = (1019s^2 + 1416s + 135, 482s^3 + 902s^2 + 1241s + 516)$
$P_7 = (1730s^2 + 1212s + 1535, 952s^3 + 258s^2 + 1205s + 579)$
$P_8 = (1141s^2 + 552s + 477, 512s^3 + 1109s^2 + 203s + 350)$

For each $P_i = (x(s), y(s)) \in E(\mathbb{F}_{1861}(s))$, we have

$$\begin{aligned} Q_i &= (t^2 x(t + 1/t), t^3 y(t + 1/t)) , \\ R_i &= (t^2 x(\zeta_5 t + 1/\zeta_5 t), t^3 y(\zeta_5 t + 1/\zeta_5 t)) \in C_{10}(\mathbb{F}_{1861}(t)) . \end{aligned}$$

So we can get 16 points of $C_{10}(\mathbb{F}_{1861}(t))$ from 8 generators of $E(\mathbb{F}_{1861}(t))$. (We use $\zeta_5 = 739$.)

$Q_1 = (899t^4 + 1845t^3 + 1103t^2 + 1845t + 899,$
$\qquad 1355t^6 + 1395t^5 + 654t^4 + 787t^3 + 654t^2 + 1395t + 1355)$
$Q_2 = (1028t^4 + 247t^3 + 594t^2 + 247t + 1028,$
$\qquad 1048t^6 + 300t^5 + 1056t^4 + 1807t^3 + 1056t^2 + 300t + 1048)$
$Q_3 = (871t^4 + 552t^3 + 1243t^2 + 552t + 871,$
$\qquad 1784t^6 + 323t^5 + 224t^4 + 1364t^3 + 224t^2 + 323t + 1784)$
$Q_4 = (1727t^4 + 834t^3 + 91t^2 + 834t + 1727,$
$\qquad 1848t^6 + 1164t^5 + 1650t^4 + 1627t^3 + 1650t^2 + 1164t + 1848)$

$$
\begin{aligned}
Q_5 = (&524t^4 + 618t^3 + 1572t^2 + 618t + 524,\\
&674t^6 + 1853t^5 + 1517t^4 + 658t^3 + 1517t^2 + 1853t + 674)\\
Q_6 = (&1019t^4 + 1416t^3 + 312t^2 + 1416t + 1019,\\
&482t^6 + 902t^5 + 826t^4 + 459t^3 + 826t^2 + 902t + 482)\\
Q_7 = (&1730t^4 + 1212t^3 + 1273t^2 + 1212t + 1730,\\
&952t^6 + 258t^5 + 339t^4 + 1095t^3 + 339t^2 + 258t + 952)\\
Q_8 = (&1141t^4 + 552t^3 + 898t^2 + 552t + 1141,\\
&512t^6 + 1109t^5 + 1739t^4 + 707t^3 + 1739t^2 + 1109t + 512)\\
R_1 = (&1203t^4 + 1203t^3 + 1103t^2 + 899t + 1320,\\
&1619t^6 + 1225t^5 + 1307t^4 + 787t^3 + 706t^2 + 829t + 803)\\
R_2 = (&796t^4 + 155t^3 + 594t^2 + 1126t + 168,\\
&1634t^6 + 1304t^5 + 625t^4 + 1807t^3 + 218t^2 + 1519t + 1007)\\
R_3 = (&1652t^4 + 369t^3 + 1243t^2 + 1552t + 12,\\
&125t^6 + 337t^5 + 1768t^4 + 1364t^3 + 441t^2 + 1530t + 1700)\\
R_4 = (&1750t^4 + 335t^3 + 91t^2 + 1293t + 1716,\\
&722t^6 + 742t^5 + 395t^4 + 1627t^3 + 108t^2 + 1204t + 142)\\
R_5 = (&1434t^4 + 757t^3 + 1572t^2 + 1333t + 817,\\
&646t^6 + 660t^5 + 741t^4 + 658t^3 + 1649t^2 + 158t + 225)\\
R_6 = (&608t^4 + 542t^3 + 312t^2 + 1392t + 811,\\
&716t^6 + 25t^5 + 6t^4 + 459t^3 + 812t^2 + 1726t + 1177)\\
R_7 = (&572t^4 + 527t^3 + 1273t^2 + 1223t + 261,\\
&1669t^6 + 1047t^5 + 1147t^4 + 1095t^3 + 144t^2 + 1418t + 1483)\\
R_8 = (&1709t^4 + 369t^3 + 898t^2 + 1552t + 1193,\\
&1054t^6 + 627t^5 + 1031t^4 + 707t^3 + 574t^2 + 1825t + 563)
\end{aligned}
$$

Let N_0 be the sublattice of $C_{10}(\mathbb{F}_{1861}(t))$ generated by these 16 points. Calculating the height pairings of them, we have the Gram matrix:

$$
\begin{pmatrix}
4 & 0 & 0 & 0 & 0 & -2 & 0 & 2 & 0 & 0 & 0 & 0 & -1 & 1 & -1 & -1\\
0 & 4 & -2 & 2 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & -1 & 1\\
0 & -2 & 4 & -2 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1\\
0 & 2 & -2 & 4 & 0 & 0 & 2 & 0 & 0 & -1 & 0 & -2 & 0 & 1 & -2 & 1\\
0 & -2 & 0 & 0 & 4 & 0 & 2 & 0 & -1 & -1 & 0 & 0 & 0 & 1 & 0 & -1\\
-2 & 0 & -2 & 0 & 0 & 4 & -2 & -2 & 1 & 0 & 0 & 1 & 1 & -2 & 2 & 2\\
0 & 0 & 0 & 2 & 2 & -2 & 4 & 0 & -1 & -1 & 0 & -2 & 0 & 2 & -2 & -1\\
2 & 0 & 0 & 0 & 0 & -2 & 0 & 4 & -1 & 1 & -1 & 1 & -1 & 2 & -1 & -2\\
0 & 0 & 0 & 0 & -1 & 1 & -1 & -1 & 4 & 0 & 0 & 0 & 0 & -2 & 0 & 2\\
0 & 0 & 0 & -1 & -1 & 0 & -1 & 1 & 0 & 4 & -2 & 2 & -2 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -2 & 4 & -2 & 0 & -2 & 0 & 0\\
0 & -1 & 0 & -2 & 0 & 1 & -2 & 1 & 0 & 2 & -2 & 4 & 0 & 0 & 2 & 0\\
-1 & -1 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & -2 & 0 & 0 & 4 & 0 & 2 & 0\\
1 & 0 & 0 & 1 & 1 & -2 & 2 & 2 & -2 & 0 & -2 & 0 & 0 & 4 & -2 & -2\\
-1 & -1 & 0 & -2 & 0 & 2 & -2 & -1 & 0 & 0 & 0 & 2 & 2 & -2 & 4 & 0\\
-1 & 1 & -1 & 1 & -1 & 2 & -1 & -2 & 2 & 0 & 0 & 0 & 0 & -2 & 0 & 4
\end{pmatrix}
$$

The determinant is

(4.17) $$\det N_0 = 5^4 .$$

If we assume

(4.18) $$C_{10}(\overline{\mathbb{F}_{1861}}(t)) \cong C_{10}(k(t)) ,$$

then

$$N_0 \subset C_{10}(\mathbb{F}_{1861}(t)) \subset C_{10}(\overline{\mathbb{F}_{1861}}(t)) \cong C_{10}(k(t)) .$$

By (4.13) and (4.17), we have

(4.19) $$N_0 = C_{10}(\overline{\mathbb{F}_{1861}}(t)) .$$

So these 16 points are the generators of $C_{10}(\overline{\mathbb{F}_{1861}}(t))$ and they belong to $C_{10}(\mathbb{F}_{1861}(t))$.

The sublattice $L_{12}(5)$

L_{12} has a sublattice $\widetilde{M_2} \oplus L_6(2) \oplus M_2$ of finite index.

First, we study M_2 in the same way as [U2, section 3]. See also [S2, section 6]. Note that C_2 in [U2] is defined by (5.10) and that it is slightly different from C_2 which we treat here.

The points of the form:

$$P = (x(t), y(t)) = (g - t, h + ct) \quad g, h, c \in k$$

generate the sublattice of index 2 in $C_2(k(t)) \cong D_4^*$. If we let $u = \mathrm{sp}_{t=0}(P) = g/h$, then u satisfy the fundamental algebraic equation:

(4.20) $$u^8 - 6u^4 - 3 = 0$$

and

(4.21) $$g = u^{-2} , \quad h = u^{-3} , \quad c = (1 - 3g^2)/2h .$$

The points

$$(x(t^6)/t^2, y(t^6)/t^3) = (g/t^2 - t^4, h/t^3 + ct^3)$$

generate a sublattice of index 2 in M_2. We denote this sublattice by N_2. Then the points

$$(t^6 x(1/t^6), t^9 y(1/t^6)) = (gt^6 - 1, ht^9 + ct^3)$$

generate $\widetilde{N_2}$, which is a sublattice of index 2 in $\widetilde{M_2}$. If (4.20) can be solved in $\mathbb{F}_p$, then all the coefficients of the generators of N_2 and $\widetilde{N_2}$ belong to $\mathbb{F}_p$.

$\widetilde{N_2} \oplus L_6(2) \oplus N_2$ is a sublattice of finite index in L_{12} and $\widetilde{N_2(5)} \oplus L_6(10) \oplus N_2(5)$ is a sublattice of finite index in $L_{12}(5)$. Here

$$N_2(5) = \{(x(t^5), y(t^5)) \mid (x(t), y(t)) \in N_2\} .$$

Next, we study L_6. The splitting field of

$$E_-^{(6)} : y^2 = x^3 + t^6 - 1$$

is $\mathbb{Q}(\zeta_{12}, \sqrt[3]{2})$ ([S4, Theorem 1]). So if the equations

$$u^{12} - 1 = 0 \tag{4.22}$$

and

$$u^3 - 2 = 0 \tag{4.23}$$

can be solved in $\mathbb{F}_p$, then we have

$$E_-^{(6)}(\overline{\mathbb{F}_p}(t)) = E_-^{(6)}(\mathbb{F}_p(t)) . \tag{4.24}$$

Since there are 12-th roots of 1 in $\mathbb{F}_p$ (see (5.5)),

$$E^{(6)}(\overline{\mathbb{F}_p}(t)) = E^{(6)}(\mathbb{F}_p(t)) . \tag{4.25}$$

The minimal prime number p such that the equations (4.15),(4.16),(4.20),(4.22) and (4.23) can be solved in $\mathbb{F}_p$ is 63901. We have the following Proposition.

PROPOSITION 8. *For all minimal sections $P \in E^{(60)}(\overline{\mathbb{F}_{63901}(t)})$ of norm* 20, $P \in E^{(60)}(\mathbb{F}_{63901}(t))$.

Proof. For L_{60} and its sublattices $L_5(12)$, $N_2(5)$,..., we use the same notation when we replace the constant field k by $\mathbb{F}_{63901}$. Let $P \in E^{(60)}(\mathbb{F}_{63901}(t))$ be a minimal section. Since the minimal norm of $L_5(12)$ and $\widetilde{L_5(12)}$ is 24, P must be in $\overline{L_{12}(5) \oplus M_{10}}$. Since $\widetilde{N_2(5)} \oplus L_6(10) \oplus N_2(5) \oplus M_{10}$ is a sublattice of finite index in $\overline{L_{12}(5) \oplus M_{10}}$, there exist a natural number m, such that

$$mP \in \widetilde{N_2(5)} \oplus L_6(10) \oplus N_2(5) \oplus M_{10} .$$

Let $\alpha \in \mathrm{Gal}(\overline{\mathbb{F}_{63901}}/\mathbb{F}_{63901})$. As we have seen, all generators of $\widetilde{N_2(5)} \oplus L_6(10) \oplus N_2(5) \oplus M_{10}$ belong to $E^{(60)}(\mathbb{F}_{63901}(t))$. So we have

$$(mP)^\alpha = mP .$$

$$m(P^\alpha - P) = O .$$

Since $E^{(60)}(\overline{\mathbb{F}_{63901}}(t))$ is torsion-free,

$$P^\alpha = P .$$

This means that $P \in E^{(60)}(\mathbb{F}_{63901}(t))$. *q.e.d.*

Theoretically, we can get all minimal sections in the same way as [U2]. But we need too much time to do it.

5. The lattices $L_9^{(433)}$, $L_{12}^{(397)}$ and $L_{18}^{(433)}$

In [U2], we use

$$E_-^{(m)} : y^2 = x^3 + t^m - 1$$

in place of

$$E^{(m)} : y^2 = x^3 + t^m + 1 .$$

Since k is an algebraically closed field,

$$E_-^{(m)}(k(t)) \cong E^{(m)}(k(t))$$

as lattices. If we define

$$L_m^{(p)} := E_-^{(m)}(\overline{\mathbb{F}_p}(t))\,, \tag{5.1}$$

then we have

$$\begin{aligned} L_9^{(433)} &= E_-^{(9)}(\mathbb{F}_{433}(t)) \quad &&([\text{U2, Proposition 1}]) \\ L_{12}^{(397)} &= E_-^{(12)}(\mathbb{F}_{397}(t)) \quad &&([\text{U2, Proposition 3}]) \\ L_{18}^{(433)} &= E_-^{(18)}(\mathbb{F}_{433}(t)) \quad &&([\text{U2, Proposition 5}]) \end{aligned} \tag{5.2}$$

Taking the height pairings with independent points, we can treat all points as vectors. We have explicit generators of $L_9^{(433)}$ and $L_{12}^{(397)}$ ([U2, Theorem 2, 3]).

According to Shioda, it is known that

$$L_9^{(433)} \cong L_9\,, \quad L_{12}^{(397)} \cong L_{12}\,, \quad L_{18}^{(433)} \cong L_{18} \tag{5.3}$$

as lattices.

Let a be a primitive $2m$-th root of 1. We can define an isomorphism as lattices

$$f : E_-^{(m)}(k(t)) \longrightarrow E^{(m)}(k(t)) \tag{5.4}$$

by

$$P = (x(t), y(t)) \longmapsto f(P) = (-x(at), \sqrt{-1}y(at))\,. \tag{5.5}$$

In the case $m = 6n$, $\widetilde{f(P)}$ is defined by (1.1)

$$\widetilde{f(P)} = (-t^{2n}x(a/t), \sqrt{-1}t^{3n}y(a/t))\,.$$

Then we have

$$f^{-1}\widetilde{f(P)} = ((t/a)^{2n}x(a^2/t), (t/a)^{3n}y(a^2/t))\,.$$

So, for $P = (x(t), y(t)) \in E_-^{(6n)}(k(t))$, we define $\widetilde{P}$ by

$$\widetilde{P} = ((t/a)^{2n}x(a^2/t), (t/a)^{3n}y(a^2/t))\,. \tag{5.6}$$

Here a is a primitive $12n$-th root of 1. We use

$$a^2 = 157\,, \quad \omega = a^8 = 34 \in \mathbb{F}_{397}$$

in the case of $L_{12}^{(397)}$, and

$$a = 4\,, \quad \omega = a^{12} = 198 \in \mathbb{F}_{433}$$

in the case of $L_{18}^{(433)}$. We also use

$$\omega = 198 \in \mathbb{F}_{433}$$

in the case of $L_9^{(433)}$.

5.1. $L_9^{(433)}$

Corresponding to the sublattices $L_3(3)$ and M, we define two sublattices of $L_9^{(433)}$ by

$$L_3^{(433)}(3) = \{x(t^3), y(t^3)) \mid (x(t), y(t)) \in L_3^{(433)}\}$$
$$M^{(433)} = \{(-t^4 x(1/t^3), \sqrt{-1}t^6 y(1/t^3)) \mid (x(t), y(t)) \in C_3(\overline{\mathbb{F}_{433}(t)})\}$$

In this case, C_3 is defined by

$$C_3 : y^2 = x^3 + t^4 - t\,. \tag{5.7}$$

By Theorem 2 of [U2], the determinant of $L_9^{(433)}$ is

$$\det L_9^{(433)} = \frac{3^5}{4}\,. \tag{5.8}$$

On the other hand, the determinant of $L_3^{(433)}(3) \oplus M^{(433)}$ is

$$\det(L_3^{(433)}(3) \oplus M^{(433)}) = \det(D_4^*(3) \oplus E_6^*(3)) = \frac{1}{4} \cdot 3^4 \cdot \frac{1}{3} \cdot 3^6 = \frac{3^9}{4}\,. \tag{5.9}$$

By Proposition 1, (5.8) and (5.9), there must be two points $R_1, R_2 \in L_9^{(433)}$ such that $3R_1, 3R_2 \in L_3^{(433)}(3) \oplus M^{(433)}$ and that R_1, R_2 and $L_3^{(433)}(3) \oplus M^{(433)}$ generate $L_9^{(433)}$.

By Theorem 2 of [U2], the following 10 points generate $L_9^{(433)}$:

$$\begin{aligned}
P_1 &= (432t^3, 179)\\
P_2 &= (235t^3, 179) = \omega P_1\\
P_3 &= (432t^3 + 421, 225t^3 + 51)\\
P_4 &= (235t^3 + 222, 225t^3 + 51) = \omega P_3\\
P_5 &= (432t^3 + 200t^2 + 234t + 421, 274t^4 + 416t^3 + 401t^2 + 257t + 382)\\
P_6 &= (235t^3 + 197t^2 + t + 222, 274t^4 + 416t^3 + 401t^2 + 257t + 382) = \omega P_5\\
P_7 &= (432t^3 + 108t^2 + 239t + 362, 415t^4 + 50t^3 + 405t^2 + 21t + 354)\\
P_8 &= (235t^3 + 167t^2 + 125t + 231, 415t^4 + 50t^3 + 405t^2 + 21t + 354) = \omega P_7\\
P_9 &= (432t^3 + 70t^2 + 391t + 273, 155t^4 + 374t^3 + 15t^2 + 119t + 354)\\
P_{10} &= (235t^3 + 4t^2 + 344t + 362, 155t^4 + 374t^3 + 15t^2 + 119t + 354) = \omega P_9
\end{aligned}$$

$P_1, \dots, P_4$ generate $L_3^{(433)}(3)$.
If we let

$$\begin{aligned}
Q_1 &= (79t^4 + 83t, 282t^6 + 251t^3 + 179)\\
Q_2 &= (361t^4 + 236t, 69t^6 + 293t^3 + 179)\\
Q_3 &= (325t^4 + 92t, 379t^6 + 73t^3 + 179)
\end{aligned}$$

$Q_1, Q_2, Q_3, \omega Q_1, \omega Q_2, \omega Q_3$ generate $M^{(433)}$.

Let

$$\begin{aligned}
R_1 &= P_2 + P_5 + Q_3 + \omega Q_3\\
&= (199t^3 + 204t^2 + 296t + 421, 133t^4 + 352t^3 + 2t^2 + 351t + 382)\\
R_2 &= P_4 + P_9 - \omega Q_1 + \omega Q_2 + Q_3\\
&= (199t^3 + 72t^2 + 258t + 65, 279t^4 + 178t^3 + 192t^2 + 139t + 231)
\end{aligned}$$

We have

$$\begin{aligned}
P_5 &= R_1 - P_2 - Q_3 - \omega Q_3\\
P_6 &= R_1 + P_1 - \omega Q_1 + \omega Q_2 + Q_3\\
P_7 &= -R_1 + R_2 - P_1 - Q_1 + Q_2\\
P_8 &= -R_1 + R_2 - P_4 - \omega Q_2 + \omega Q_3\\
P_9 &= R_2 - P_4 + \omega Q_1 - \omega Q_2 - Q_3\\
P_{10} &= R_2 + P_3 - Q_1 - \omega Q_1 + Q_2
\end{aligned}$$

and

$$\begin{aligned}
3R_1 &= -P_1 + P_2 - Q_1 + \omega Q_1 + Q_2 - \omega Q_2\\
3R_2 &= -P_3 + P_4 - Q_2 + \omega Q_2 + Q_3 - \omega Q_3
\end{aligned}$$

So R_1, R_2 are two points in $L_9^{(433)}$ such that $3R_1, 3R_2 \in L_3^{(433)}(3) \oplus M^{(433)}$ and that R_1, R_2 and $L_3^{(433)}(3) \oplus M^{(433)}$ generate $L_9^{(433)}$.

5.2. $L_{12}^{(397)}$

Corresponding to the sublattices $L_4(3)$, $L_6(2)$ and M_2, we define sublattices of $L_{12}^{(397)}$ by

$$\begin{aligned}
L_4^{(397)}(3) &= \{x(t^3), y(t^3)) \mid (x(t), y(t)) \in L_4^{(397)}\}\\
L_6^{(397)}(2) &= \{x(t^2), y(t^2)) \mid (x(t), y(t)) \in L_6^{(397)}\}\\
M_2^{(397)} &= \{(x(t^6)/t^2, y(t^6))/t^3 \mid (x(t), y(t)) \in C_2(\overline{\mathbb{F}_{397}}(t))\}
\end{aligned}$$

In this case, C_2 is defined by

$$C_2 : y^2 = x^3 + t^3 - t\,. \tag{5.10}$$

By Theorem 3 of [U2], the following 16 points generate $L_{12}^{(397)}$:

$P_1 = (1, t^6)$
$P_2 = (34, t^6)$
$P_3 = (396t^4, 63)$
$P_4 = (363t^4, 63)$
$P_5 = (367t^4 + 362, 328t^6 + 236t^2)$
$P_6 = (363t^4 + 141, 179t^4 + 20)$
$P_7 = (2t^4 + 266t^2 + 30, 394t^6 + 262t^4 + 307t^2 + 377)$
$P_8 = (171t^4 + 87t^2 + 329, 328t^6 + 285t^4 + 2294t^2 + 208)$
$P_9 = (207t^3 + 59, t^6 + 183t^3 + 319)$
$P_{10} = (289t^3 + 21, t^6 + 183t^3 + 319)$
$P_{11} = (338t^4 + 190t, 247t^6 + 16t^3 + 63)$
$P_{12} = (376t^4 + 108t, 247t^6 + 16t^3 + 63)$
$P_{13} = (338t^4 + 60t, 247t^6 + 214t^3 + 334)$
$P_{14} = (342t^3 + 21, t^6 + 381t^3 + 78)$
$P_{15} = (224t^4 + 17t^3 + 297t^2 + 230t + 324,$
$\quad 281t^6 + 127t^5 + 217t^4 + 207t^3 + 297t^2 + 49t + 162)$
$P_{16} = (73t^4 + 167t^3 + 100t^2 + 380t + 173,$

$$281t^6 + 308t^5 + 52t^4 + 337t^3 + 173t^2 + 61t + 235)$$

The determinant of $L_{12}^{(397)}$ is

$$\det L_{12}^{(397)} = 2^4 3^4 . \tag{5.11}$$

$P_1, \ldots, P_8$ generate $L_6^{(397)}(2)$.

$P_1, P_2, P_9, P_{10}, P_{14}, \omega P_{14}$ generate $L_4^{(397)}(3)$.

$P_3, P_4, P_{11}, P_{12}, P_{13}, \omega P_{13}$ generate $\widetilde{L_4^{(397)}}(3)$.

$P_1, \ldots, P_{14}, \omega P_{13}, \omega P_{14}$ generate $L_4^{(397)}(3) + L_6^{(397)}(2) + \widetilde{L_4^{(397)}}(3)$.

The determinant of $L_4^{(397)}(3) + L_6^{(397)}(2) + \widetilde{L_4^{(397)}}(3)$ is

$$\det(L_4^{(397)}(3) + L_6^{(397)}(2) + \widetilde{L_4^{(397)}}(3)) = 2^4 3^8 . \tag{5.12}$$

By (4.8), we have

$$L_{12}^{(397)} = L_4^{(397)}(3) + L_6^{(397)}(2) + \widetilde{L_4^{(397)}}(3) + \overline{M_2^{(397)} \oplus \widetilde{M_2^{(397)}}} . \tag{5.13}$$

By (5.11), (5.12) and (5.13), there must be two points $R_1, R_2 \in \overline{M_2^{(397)} \oplus \widetilde{M_2^{(397)}}}$ such that

$$3R_1, 3R_2 \in L_4^{(397)}(3) + L_6^{(397)}(2) + \widetilde{L_4^{(397)}}(3)$$

and that R_1, R_2 and $L_4^{(397)}(3) + L_6^{(397)}(2) + \widetilde{L_4^{(397)}}(3)$ generate $L_{12}^{(397)}$.
Let

$$\begin{aligned} R_1 &= (35t^4 + 206t^2 + 34, 288t^5 + 298t^3 + 155t) \\ R_2 &= (35t^4 + 191t^2 + 34, 118t^5 + 115t^3 + 160t) \end{aligned} \tag{5.14}$$

We have

$$\begin{aligned} P_{15} &= R_1 - P_1 - P_2 + P_9 - P_{11} - P_{12} \\ P_{16} &= R_2 - P_4 + P_9 + P_{11} + P_{12} + P_{14} - \omega P_{13} \end{aligned} \tag{5.15}$$

If we let

$$\begin{aligned} Q_1 &= (137t^6 + 362, 145t^9 + 312t^3) \in \widetilde{M_2^{(397)}} \\ Q_2 &= (260t^6 + 362, 393t^9 + 203t^3) \in \widetilde{M_2^{(397)}} \end{aligned} \tag{5.16}$$

then we have

$$\begin{aligned} 3R_1 &= Q_1 - \omega Q_1 + \widetilde{Q_1} - \omega \widetilde{Q_1} \\ 3R_2 &= Q_2 - \omega Q_2 + \widetilde{Q_2} - \omega \widetilde{Q_2} \end{aligned} \tag{5.17}$$

So R_1 and R_2 are two points of $L_{12}^{(397)}$ such that $3R_1, 3R_2 \in M_2^{(397)} \oplus \widetilde{M_2^{(397)}}$ and that R_1, R_2 and $L_4^{(397)}(3) + L_6^{(397)}(2) + \widetilde{L_4^{(397)}}(3)$ generate $L_{12}^{(397)}$.

By (5.17) and (5.14), we can see that R_1 and R_2 satisfy

$$\widetilde{R_i} = R_i \quad \text{and} \quad R_i^{\tau} = -R_i \quad (i = 1, 2) . \tag{5.18}$$

So if we define a sublattice $T^{(397)}$ of $L_{12}^{(397)}$ by

(5.19) $$T^{(397)} = \{P \in L_{12}^{(397)} \mid \widetilde{P} = P \quad \text{and} \quad P^{\tau} = -P\}\,,$$

then we have

(5.20) $$L_{12}^{(397)} = L_4^{(397)}(3) + L_6^{(397)}(2) + \widetilde{L_4^{(397)}}(3) + T^{(397)}\,.$$

The corresponding sublattice

(5.21) $$T = \{P \in L_{12} \mid \widetilde{P} = P \quad \text{and} \quad P^{\tau} = -P\}$$

of L_{12} is described by

(5.22) $$T = \{(t^2x(t^2+1/t^2), t^3y(t^2+1/t^2)) \mid (x(s), y(s)) \in F(k(s))\}\,.$$

Here F is the elliptic curve over $k(s)$ defined by

(5.23) $$F : y^2 = x^3 + s^3 - 3s\,.$$

5.3. $L_{18}^{(433)}$

Corresponding to the sublattices $L_9(2)$, $L_6(3)$ and $L_3(6)$, we define sublattices of $L_{18}^{(433)}$ by

$$L_9^{(433)}(2) = \{x(t^2), y(t^2)) \mid (x(t), y(t)) \in L_9^{(433)}\}$$
$$L_6^{(433)}(3) = \{x(t^3), y(t^3)) \mid (x(t), y(t)) \in L_6^{(433)}\}$$
$$L_3^{(433)}(6) = \{x(t^6), y(t^6)) \mid (x(t), y(t)) \in L_3^{(433)}\}$$

By Theorem 1, we have the following Corollary.

COROLLARY 1. *If we assume* $L_{18}^{(433)} \cong L_{18}$ (*see* (5.3)), *then* L_{18} *has* 672 *minimal sections of norm 6 and the determinant is* $2^{12}3^{10}$.

Proof. By Corollary 1 of [U2], $L_9(2)$ has 240 minimal sections of norm 6 and $\widetilde{L_9(2)}$ also has 240 minimal sections. $L_6(3) \cong E_8(3)$ has 240 minimal sections. $L_9(2) \cap L_6(3) = L_3(6) \cong D_4^*(6)$ has 24 minimal sections and $\widetilde{L_9(2)} \cap L_6(3) = \widetilde{L_3(6)} \cong D_4^*(6)$ also has 24 minimal sections. So the number of minimal sections of L_{18} is $240+240+240-24-24 = 672$.

By Theorem 1, we have

$$L_{18}^{(433)} = L_9^{(433)}(2) + L_6^{(433)}(3) + \widetilde{L_9^{(433)}}(2)\,.$$

By Theorem 2 of [U2], the following 10 points generate $L_9^{(433)}(2)$:

$P_1 = (432t^6, 179)$
$P_2 = (235t^6, 179)$
$P_3 = (432t^6 + 421, 225t^6 + 51)$
$P_4 = (235t^6 + 222, 225t^6 + 51)$
$P_5 = (432t^6 + 200t^4 + 234t^2 + 421, 274t^8 + 416t^6 + 401t^4 + 257t^2 + 382)$
$P_6 = (235t^6 + 197t^4 + t^2 + 222, 274t^8 + 416t^6 + 401t^4 + 257t^2 + 382)$
$P_7 = (432t^6 + 108t^4 + 239t^2 + 362, 415t^8 + 50t^6 + 405t^4 + 21t^2 + 354)$

$$\begin{aligned}
P_8 &= (235t^6 + 167t^4 + 125t^2 + 231, 415t^8 + 50t^6 + 405t^4 + 21t^2 + 354)\\
P_9 &= (432t^6 + 70t^4 + 391t^2 + 273, 155t^8 + 374t^6 + 15t^4 + 119t^2 + 354)\\
P_{10} &= (235t^6 + 4t^4 + 344t^2 + 362, 155t^8 + 374t^6 + 15t^4 + 119t^2 + 354)
\end{aligned}$$

P_1, P_2, P_3, P_4 generate $L_3^{(433)}(6)$.
If we let

$$\begin{aligned}
Q_1 &= (253t^6 + 11t^3 + 119, 335t^9 + 6t^6 + 384t^3 + 211) \in L_6^{(433)}(3)\\
Q_2 &= \omega Q_1 = (299t^6 + 13t^3 + 180, 335t^9 + 6t^6 + 384t^3 + 211) \in L_6^{(433)}(3)
\end{aligned}$$

then Q_1 and Q_2 satisfy

$$\begin{aligned}
2Q_1 &= P_1 + P_3 + \widetilde{P}_1 + \widetilde{P}_3\\
2Q_2 &= P_2 + P_4 + \widetilde{P}_2 + \widetilde{P}_4
\end{aligned}$$

The determinant of the sublattice of $L_6^{(433)}(3)$ generated by $P_1, P_2, P_3, P_4, \widetilde{P}_1, \widetilde{P}_2, Q_1, Q_2$ is 3^8. So this is $L_6^{(433)}(3)$.

$P_1, \ldots, P_{10}, \widetilde{P}_1, \widetilde{P}_2, Q_1, Q_2, \widetilde{P}_5, \ldots, \widetilde{P_{10}}$, generate $L_9^{(433)}(2) + L_6^{(433)}(3) + \widetilde{L_9^{(433)}}(2)$.
The Gram matrix is

$$\begin{pmatrix}
6 & -3 & 0 & 3 & 0 & 3 & -2 & 1 & -2 & 1 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
-3 & 6 & -3 & 0 & -3 & 0 & 1 & -2 & 1 & -2 & 0 & 0 & -3 & 3 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & -3 & 6 & -3 & 2 & -1 & -2 & 1 & 0 & 3 & 0 & 0 & 3 & -3 & 0 & 0 & 0 & 0 & 0 & 0\\
3 & 0 & -3 & 6 & -1 & 2 & 1 & -2 & -3 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & -3 & 2 & -1 & 6 & -3 & 0 & -1 & 2 & 1 & 0 & 0 & 1 & -2 & 0 & 0 & 0 & 0 & 0 & 0\\
3 & 0 & -1 & 2 & -3 & 6 & 1 & 0 & -3 & 2 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
-2 & 1 & -2 & 1 & 0 & 1 & 6 & -3 & 0 & 1 & 0 & 0 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & -2 & 1 & -2 & -1 & 0 & -3 & 6 & -1 & 0 & 0 & 0 & 1 & -2 & 0 & 0 & 0 & 0 & 0 & 0\\
-2 & 1 & 0 & -3 & 2 & -3 & 0 & -1 & 6 & -3 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & -2 & 3 & 0 & 1 & 2 & 1 & 0 & -3 & 6 & 0 & 0 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & -3 & 3 & 0 & 0 & 3 & -2 & 1 & -2 & 1\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & 6 & -3 & 3 & -3 & 0 & 1 & -2 & 1 & -2\\
3 & -3 & 3 & 0 & 1 & 1 & -2 & 1 & -1 & 2 & 3 & -3 & 6 & -3 & 1 & 1 & -2 & 1 & -1 & 2\\
0 & 3 & -3 & 3 & -2 & 1 & 1 & -2 & -1 & -1 & 0 & 3 & -3 & 6 & -2 & 1 & 1 & -2 & -1 & -1\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & 1 & -2 & 6 & -3 & 0 & -1 & 2 & 1\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 1 & 1 & -3 & 6 & 1 & 0 & -3 & 2\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 1 & -2 & 1 & 0 & 1 & 6 & -3 & 0 & 1\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & -2 & -1 & 0 & -3 & 6 & -1 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 1 & -1 & -1 & 2 & -3 & 0 & -1 & 6 & -3\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 2 & -1 & 1 & 2 & 1 & 0 & -3 & 6
\end{pmatrix}$$

The determinant is $2^{12}3^{10}$. *q.e.d.*

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